Mixnets for Voting

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Outline

Goal of the Talk (and Disclaimer)

- Our goal is to give a conceptually simple description of mix-nets.
- We make no attempt to give fair historical account.
- We also ignore many variants and extensions of mix-nets. In particular, those that do not have applications in voting.

Simulation Paradigm

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- The ideal functionality must be **secure by inspection**.
- Show that:
  for every **adversary** interacting with the real protocol,
  there is a **simulator** interacting with the ideal functionality,
  such that no outside **distinguisher** can distinguish:
  - the adversary and real protocol from
  - the simulator and ideal functionality.

General Security Model

- **Efficient Adversary.** The adversary must run in polynomial time.
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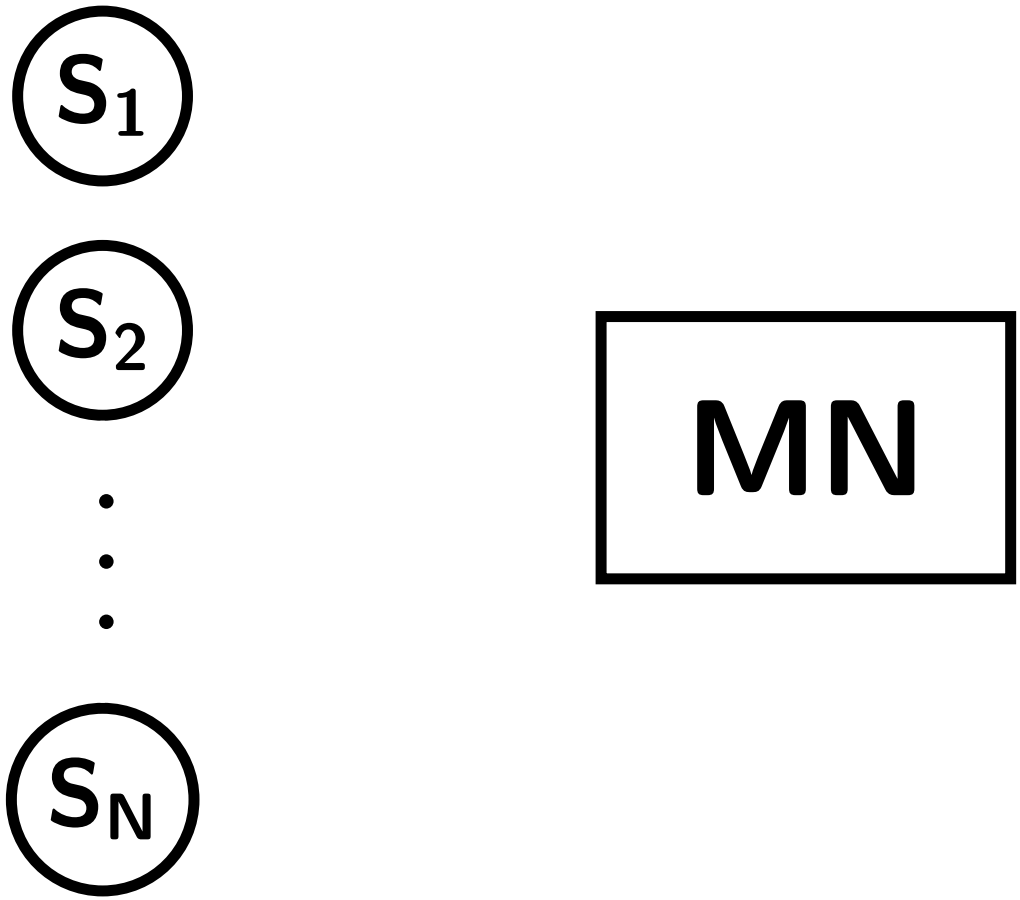
- **Scheduling.** In this talk we ignore the problem of corruption of the network. The results are mostly independent of this.

Mix-Net Basics

Ideal Mix-Net (1/5)

- There are N senders, $S_1, \ldots, S_N$.
- Each S_i wants to send/publish a message m_i.
- The senders wish to keep secret who sent/published which message.

Ideal Mix-Net (2/5)



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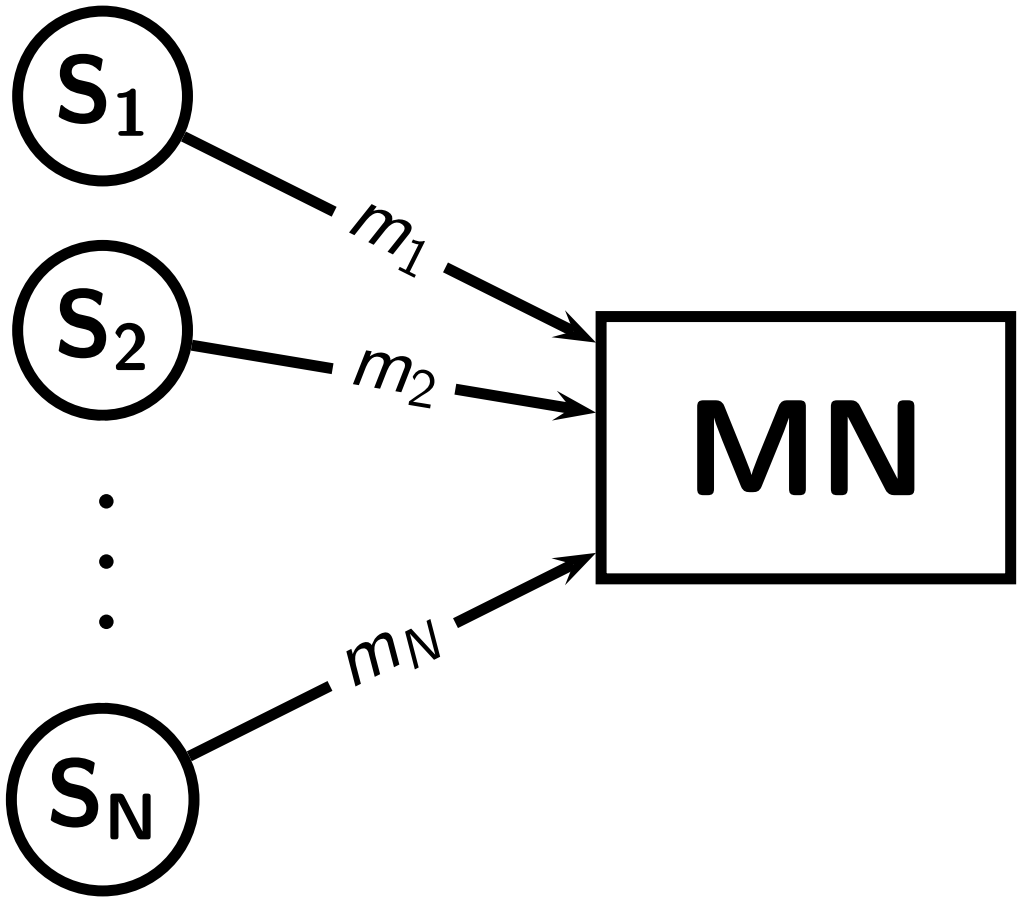

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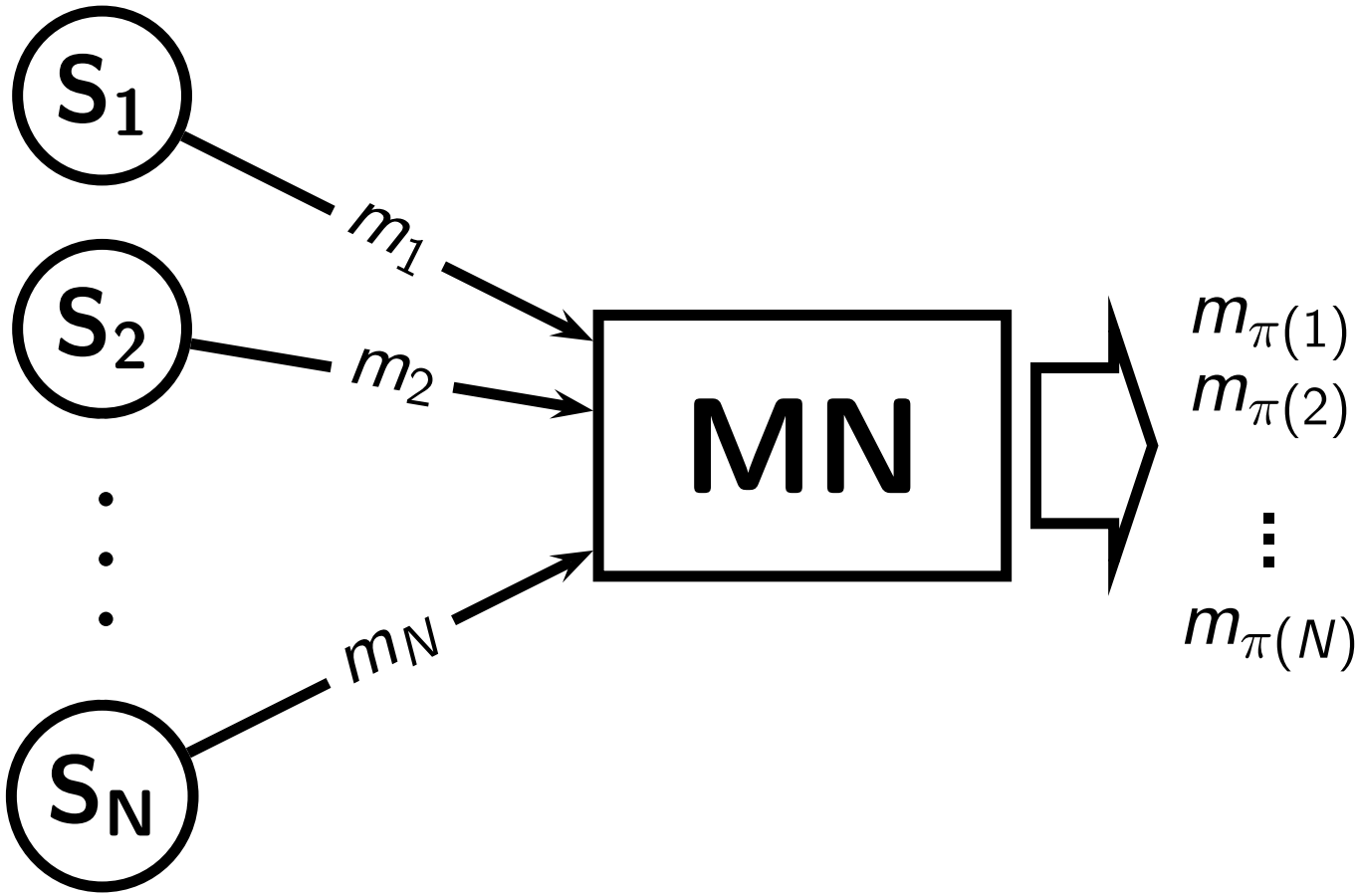

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Ideal Mix-Net (3/5)

How many messages can be input by each sender?

- Typically, each sender may submit at most one message when a mix-net is used in an election.
- In this talk we assume that each sender submits at **exactly** one message.

Ideal Mix-Net (4/5)

Who decides when the mix-net produces the output?

- A group of authorities formally make the decision that an election is closed and this typically takes place at a pre-determined time.
- In this talk, we assume that the mix-net produces its output when it has received exactly one message from each sender.
- In a real mix-net the operators decide.

Ideal Mix-Net (5/5)

Which senders can be corrupted?

- We assume that the adversary can corrupt any sender it wants.
- We can obviously not provide any anonymity for corrupted senders. They **are** the adversary.
- Privacy must hold within the set of non-corrupted senders.

Real Mix-Nets

- An (almost) ideal mix-net is what we want in the end.
- A protocol that implements an ideal mixnet is called a **mix-net** (or **mix**, **anonymizer**, etc).
- The parties that execute the protocol (along with the senders) are called **mix-servers** (or **mix-centers**, **mixers**, etc).

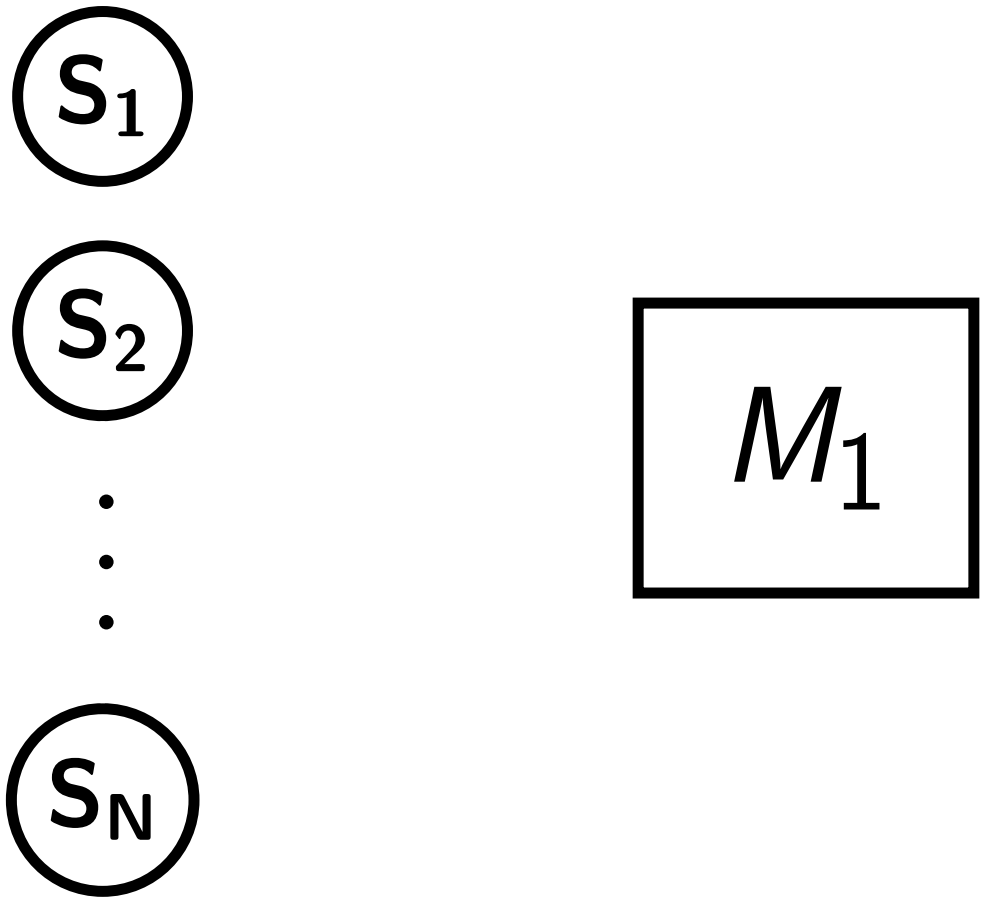
Notation

- $M_1, \ldots, M_k$ are mix-servers, **with** k **small**.
- $S_1, \ldots, S_N$ are senders, **with** N **large**.

Question

How do we implement a mix-net using a single trusted mix-server in a provably secure way?

Single Trusted Mix-Server (1/3)



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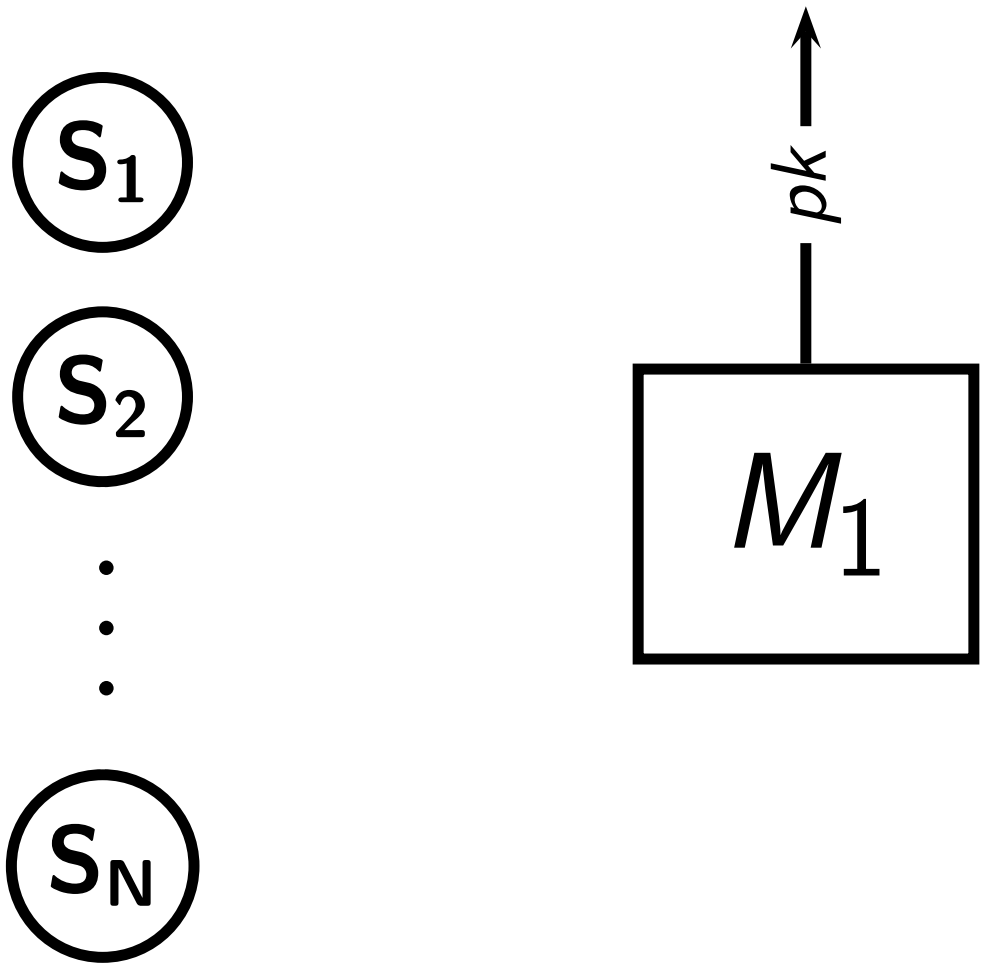

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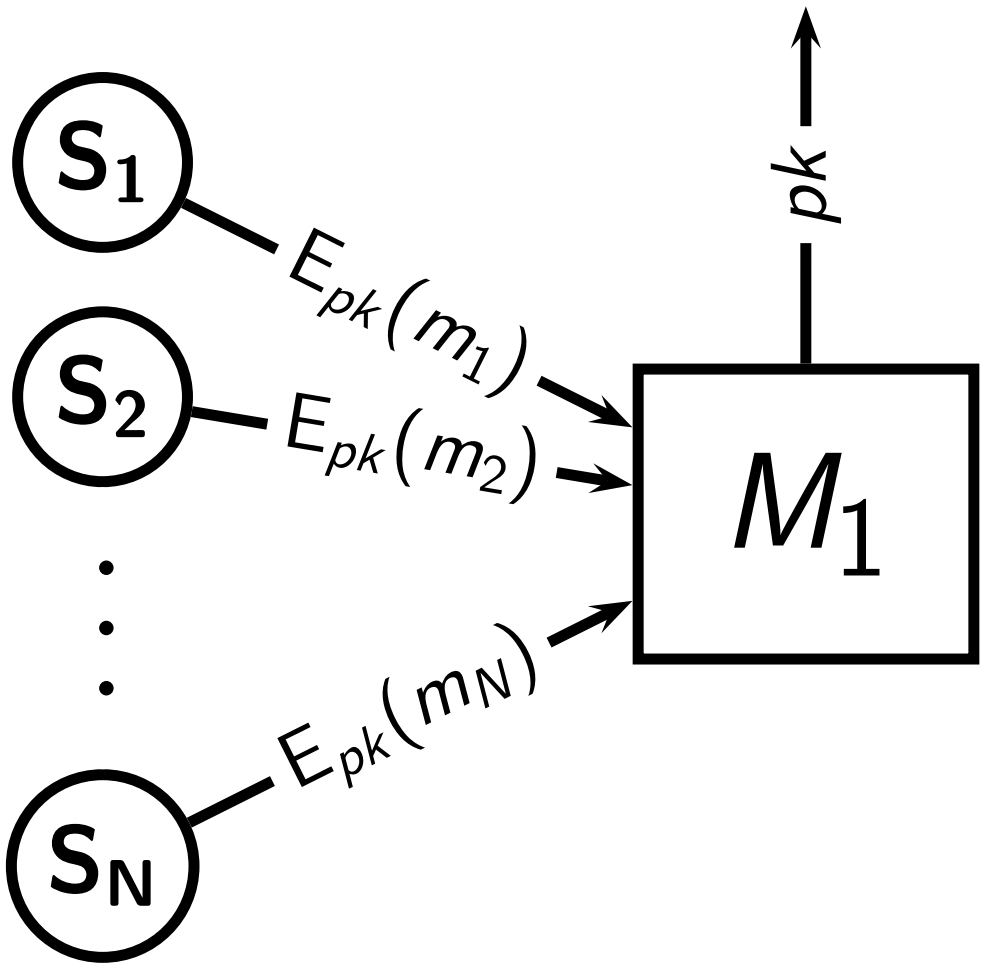

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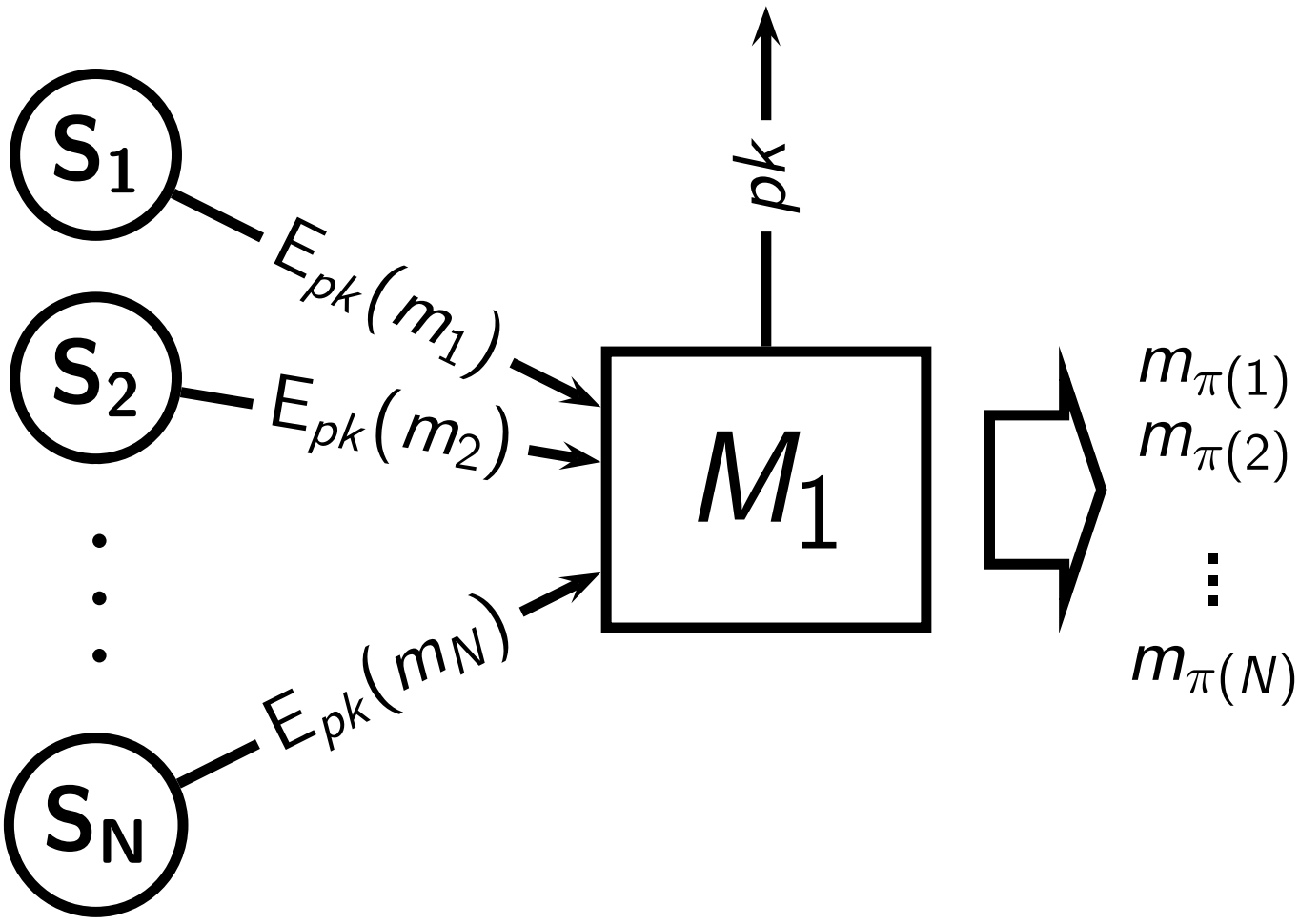

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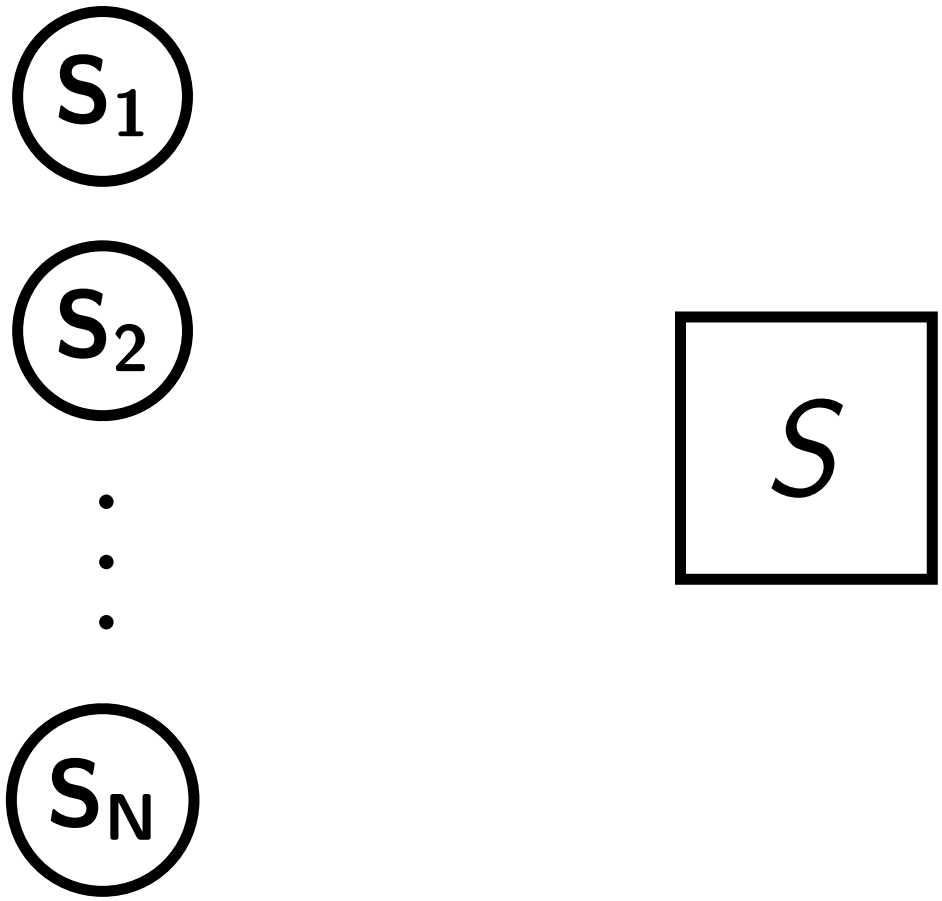

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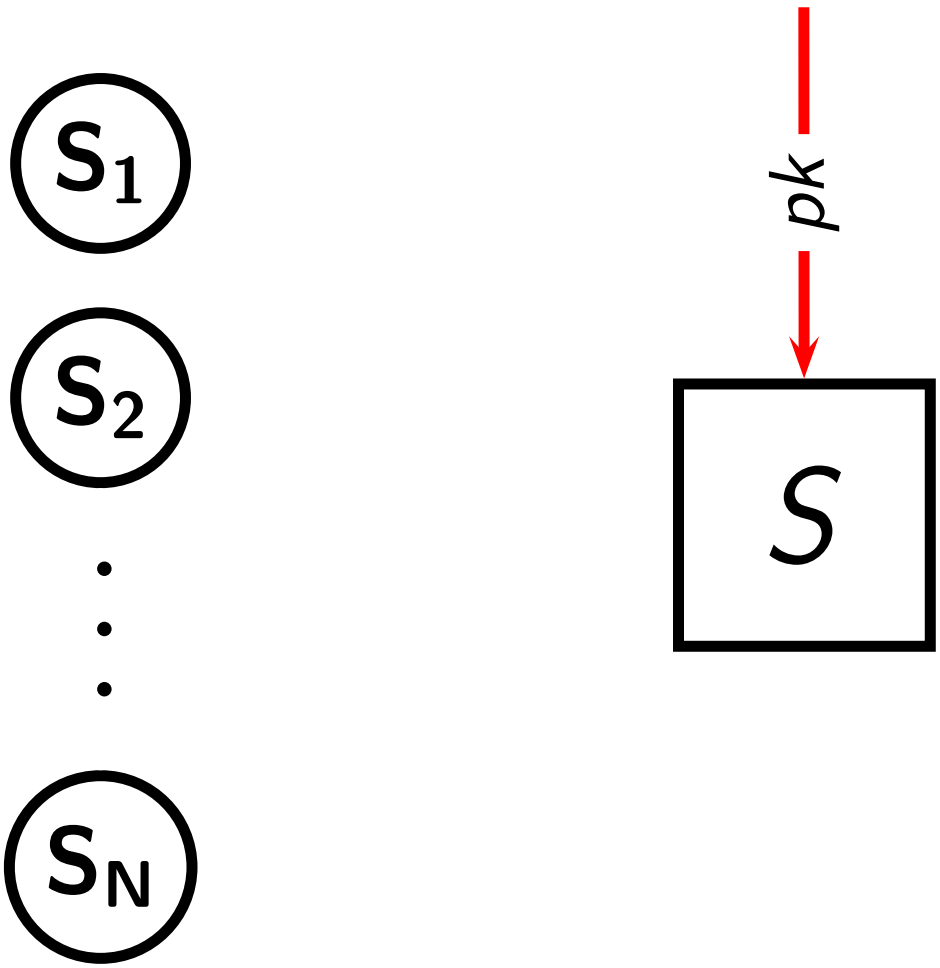

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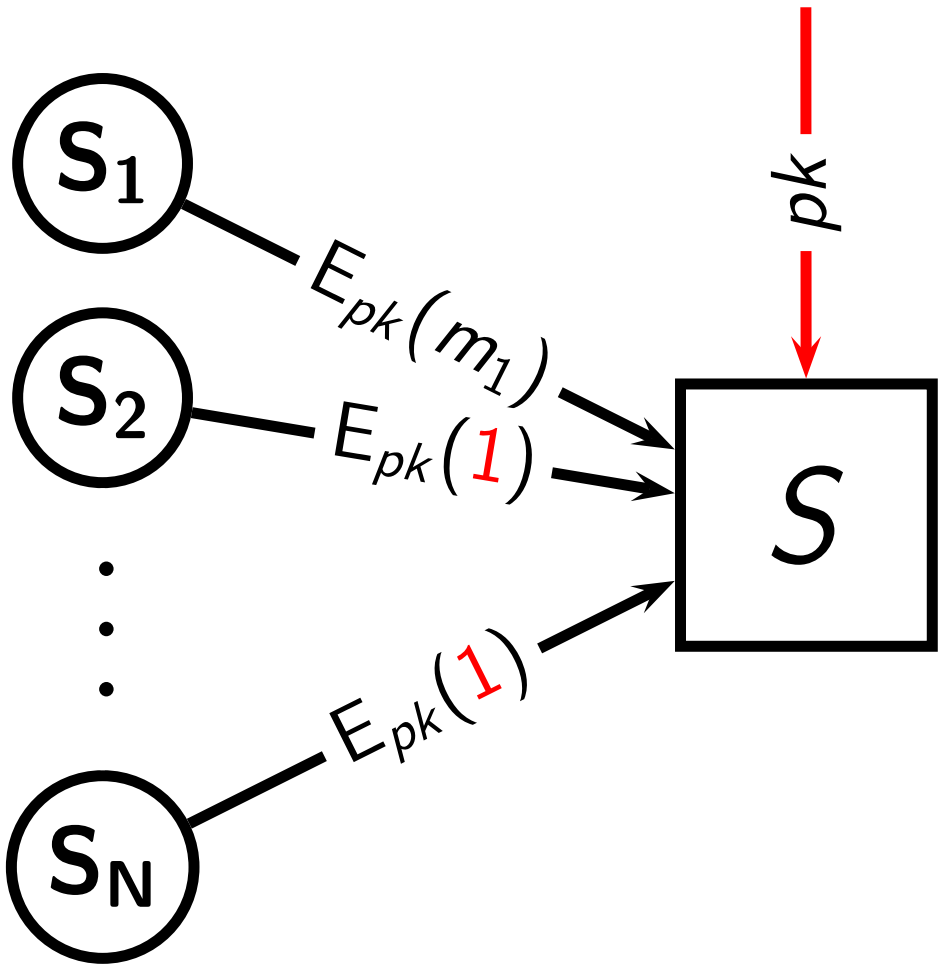

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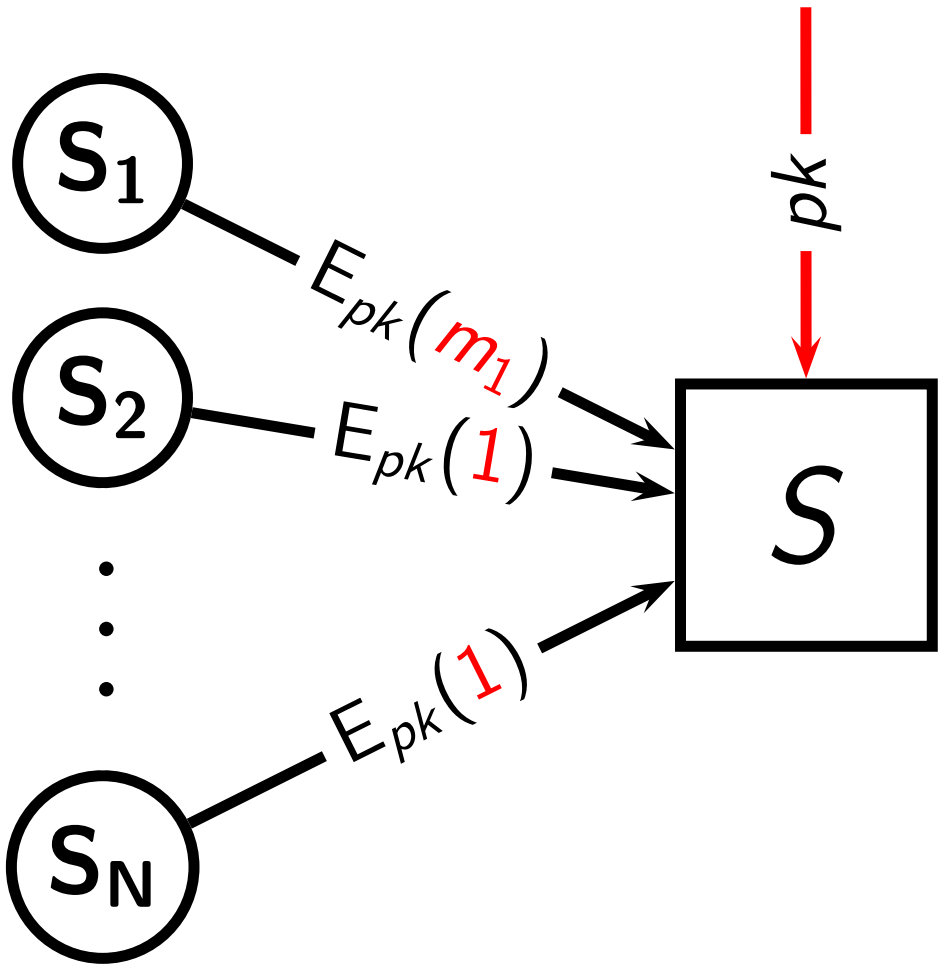

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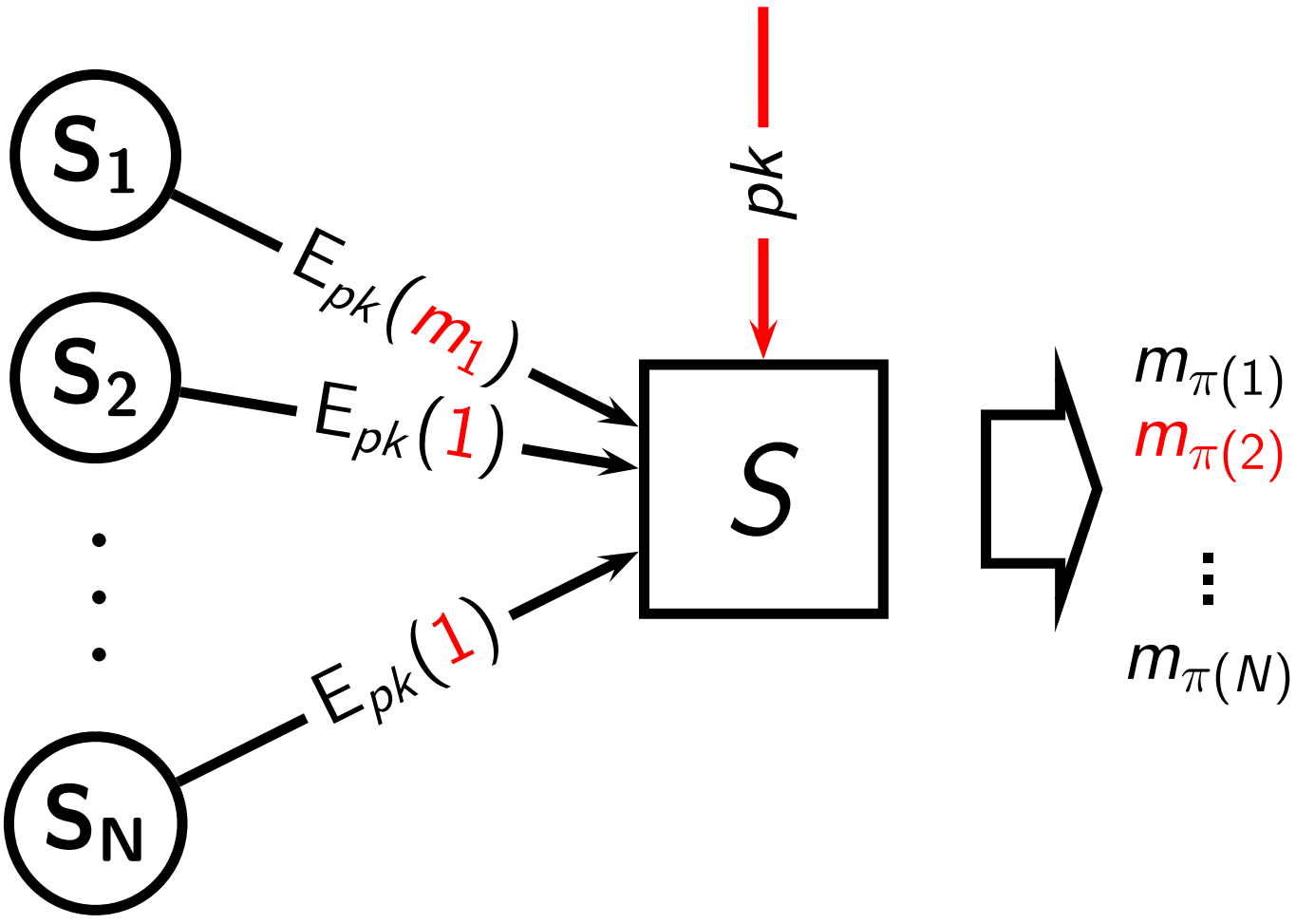

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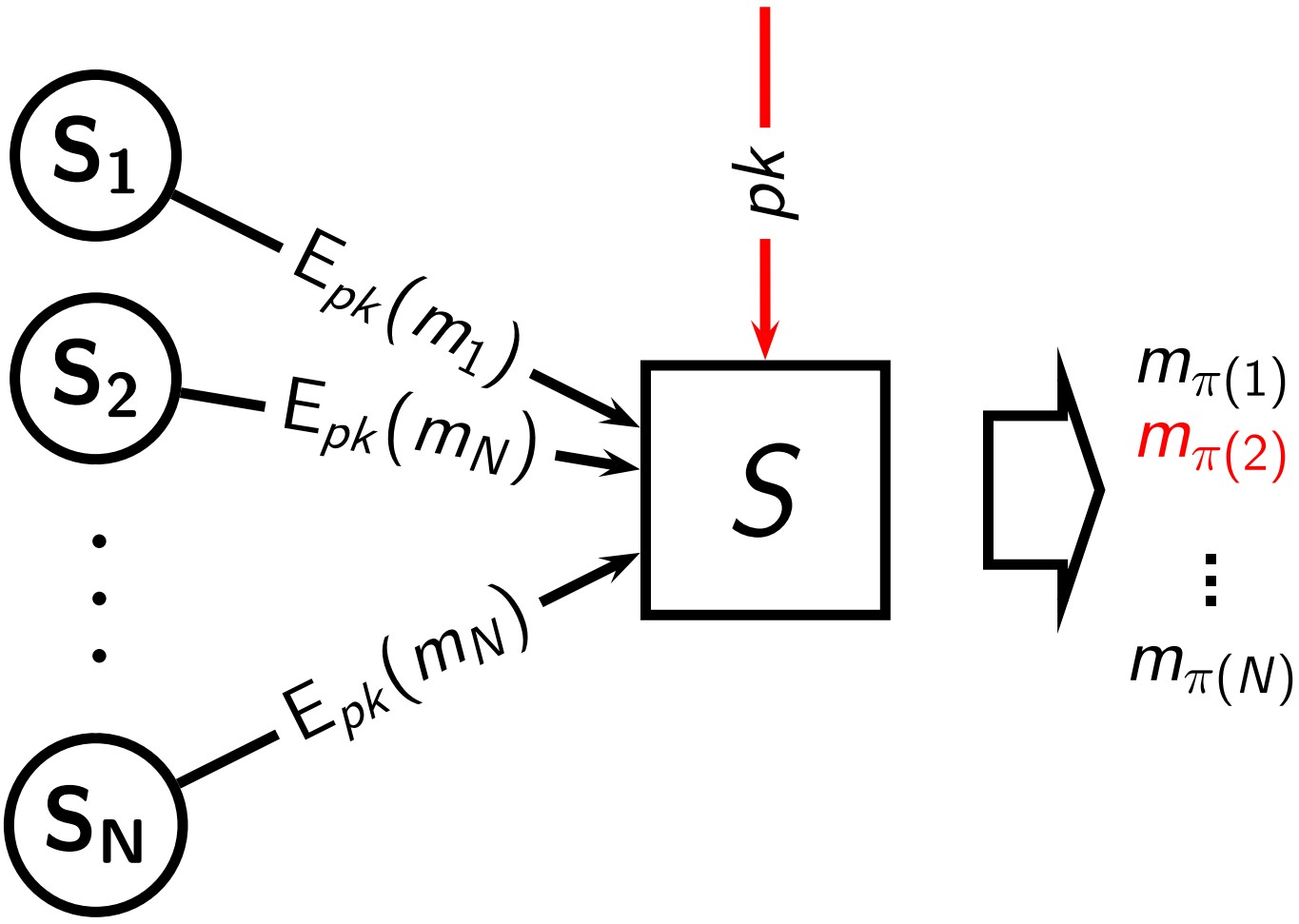

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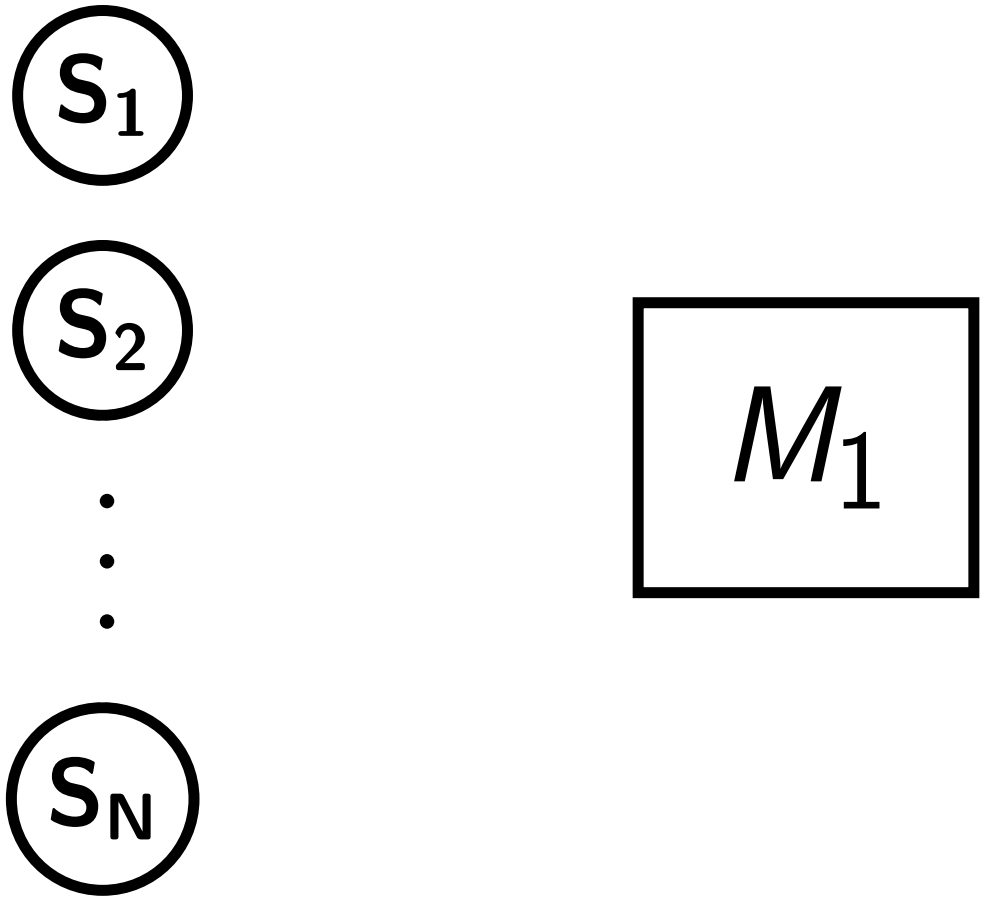

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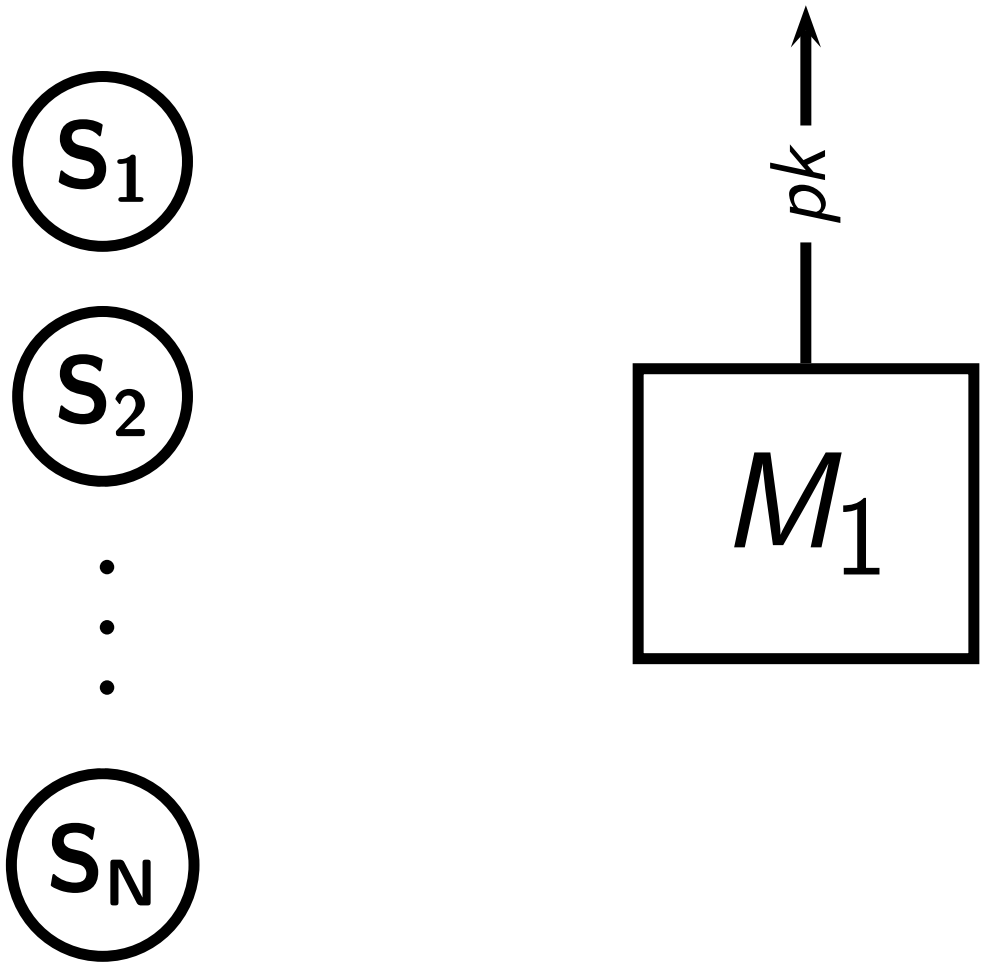

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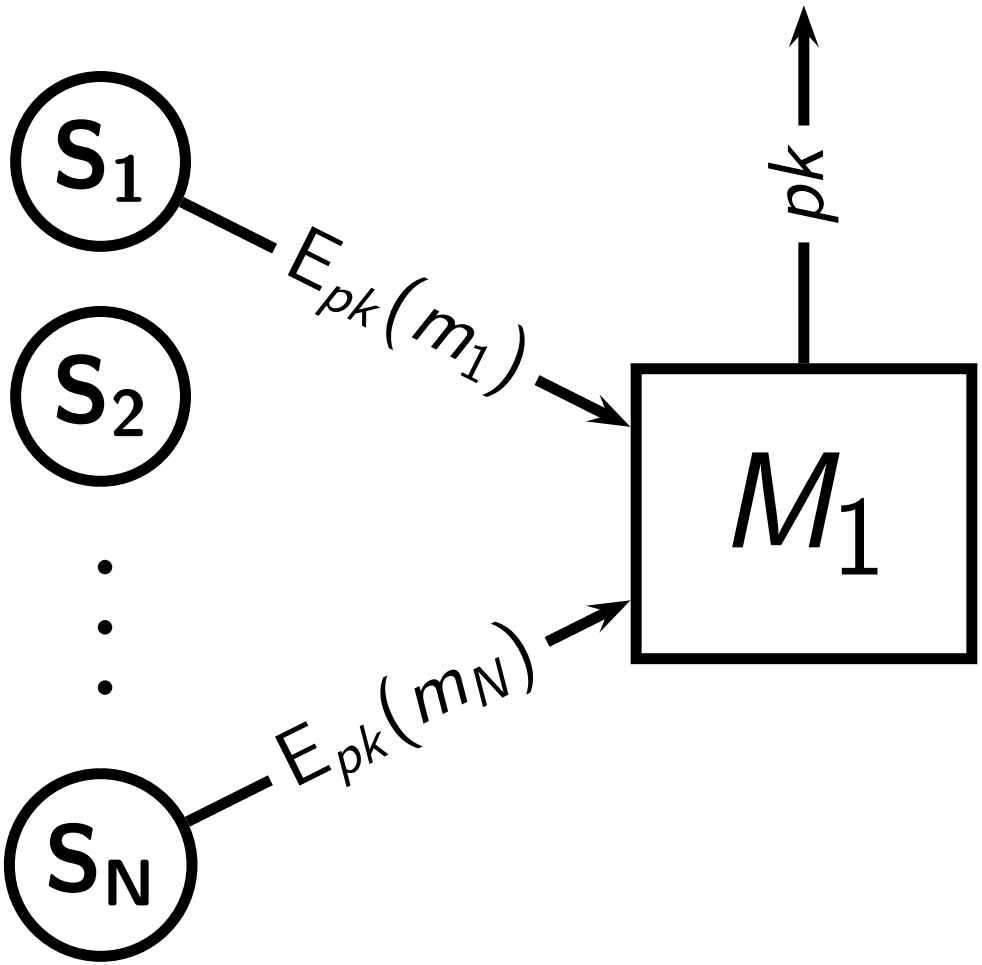

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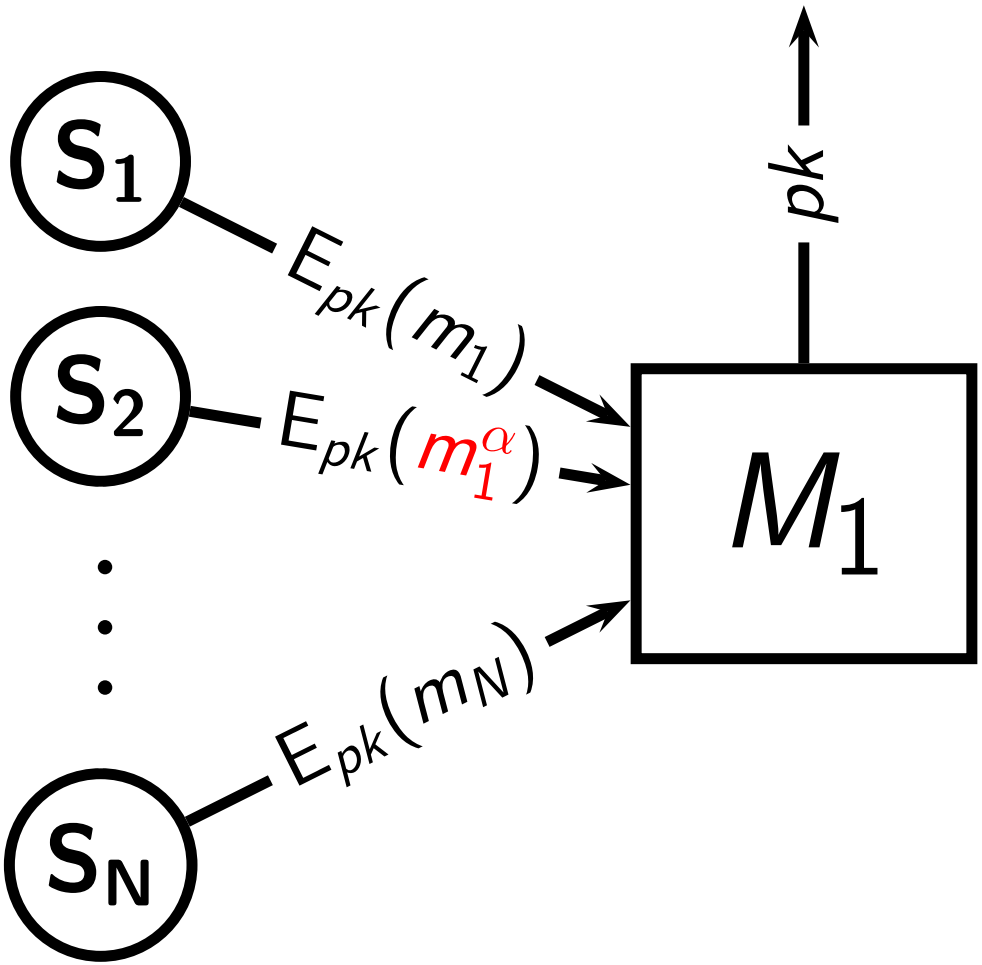

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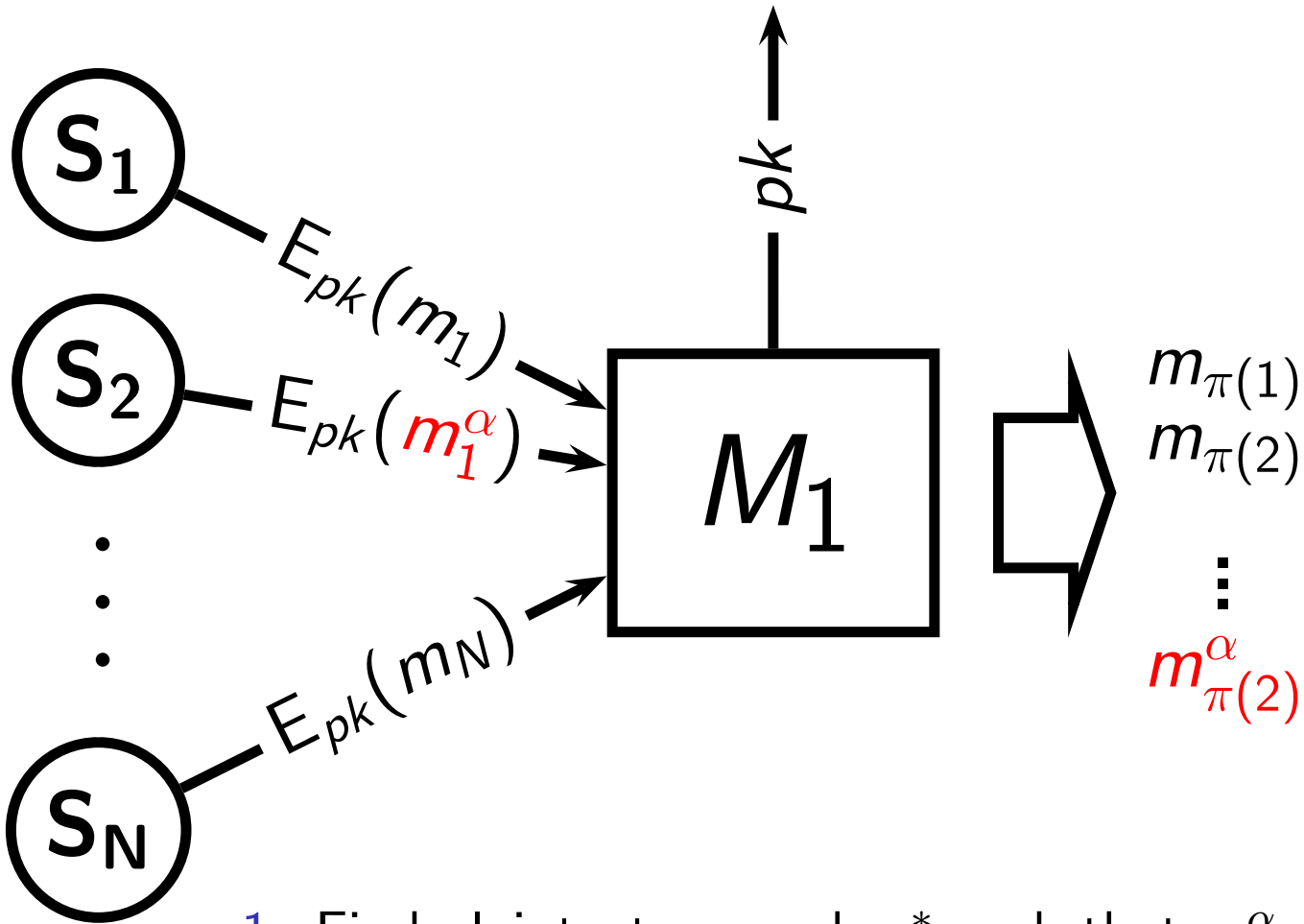

1. Find plaintexts m and m^* such that $m^\alpha = m^*$.
2. Very likely that $m_1 = m$!

Chaum's Question

Can we really trust a single mix-server?

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- Use several mix-servers $M_1, \ldots, M_k$ to share trust.
- Communication takes place over over a public bulletin board.

Chaum's Mix-Net (1/2)

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4. The output of M_k is the list of plaintexts $m_1, \ldots, m_N$ in random order.

Chaum's Mix-Net (2/2)

Theorem

If the cryptosystem is semantically secure, then Chaum's mix-net is secure against **passive** *adversaries that statically corrupts all except one mix-server.*

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Repeated application of the single trusted mix-server case. □

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- We must enforce correctness using additional techniques!

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- Even with proofs of correct behavior a mix-server could prevent any output by halting!
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- We must accept a tradeoff between privacy and robustness.

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- Repeated encryption makes it hard to construct an efficient proof of knowledge of message and gives large ciphertexts.
- Repeated encryption makes it hard to jointly generate ciphertexts with known, but permuted, plaintexts as needed in, e.g., Prêt a Votér.

Homomorphic Mix-Nets

Homomorphic Cryptosystem

Messages, ciphertexts, and randomness form groups:

- M_{pk} denotes message space.
- C_{pk} denotes ciphertext space.
- R_{pk} denotes randomness space.

and encryption satisfies:

$$\mathsf{E}_{pk}(m_0, r_0) \times \mathsf{E}_{pk}(m_1, r_1) = \mathsf{E}_{pk}(m_0 \cdot m_1, r_0 + r_1)$$

Examples: El Gamal, Paillier, ...

Re-encryption

- A ciphertext $u = \mathsf{E}_{pk}(m, r)$ can be **re-encrypted** using fresh randomness $r' \in \mathsf{R}_{pk}$ using only the public key:

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- The group structure of R_{pk} implies that
  - $r' + r$ and r are identically distributed, so
  - u and u' are identically distributed.

Distributed Key-Generation and Decryption

Distributed Key Generation

- A random public key with associated verifiably secretly shared secret key can be jointly generated.
- The simulator takes a public key as input and simulates the protocol to output it.

Distributed Decryption

- A ciphertext can be jointly decrypted using the verifiably shared secret key, without recovering it.
- The simulator of the protocol takes public key, ciphertext, and plaintext as input.

Sako and Kilian's Mix-Net

1. The mix-servers jointly generate a public key pk.
2. S_i encrypts its message m_i as $\mathrm{E}_{pk}(m_i)$.
3. For $j = 1, \ldots, k$:
   - M_j re-encrypts each ciphertext output by the previous mix-server and permutes the resulting ciphertexts.
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3. For $j = 1, \ldots, k$:
   - M_j chooses $r_{j,i} \in \mathsf{R}_{pk}$ and π_j randomly and defines

     $$L_j = (u_{j,i}) \quad \text{where} \quad u_{j,i} = \mathsf{E}_{pk}(1, r_{j,\pi_j(i)}) \times u_{j-1,\pi_j(i)}$$

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Sako and Kilian's Mix-Net (contd.)

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- Proofs of shuffles of uncorrupted mix-servers are simulated.
- Permutations used by corrupt mix-servers are extracted.
- This allows correct simulation of decryption of messages sent by corrupted senders.

Proofs of Shuffles

Batch-Proof of Re-encryption

Goal. Given pk, $(u_1, \ldots, u_N)$ and $(u'_1, \ldots, u'_N)$, prove knowledge of r_i such that

$$u'_i = \mathsf{E}_{pk}(1, r_i) \times u_i$$

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Analysis

- Extract N linearly independent vectors $\overline{e}_j = (e_{j,1}, \ldots, e_{j,N})$ and t_j such that:

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&= u_l'
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Batch-Proof of Re-encryption Under Permutation [N01,FS01]

Goal. Given pk, $(u_1, \ldots, u_N)$ and $(u'_1, \ldots, u'_N)$, prove knowledge of r_i such that

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&= u_{\pi^{-1}(l)}{}'
\end{aligned}$$

What Remains?

Great, batch-proofs almost work under permutation, but how do we prove correct form of $u' = \prod_i (u'_i)^{e_{\pi(i)}}$?

Schwartz-Zippel

Theorem (Schwartz-Zippel)

Let $f \in \mathbb{Z}_q[x_1, x_2, \ldots, x_N]$ be a non-zero polynomial of degree N, let $S \subset \mathbb{Z}_q$, and let $e_i \in S$ be randomly chosen. Then

$$\Pr[f(e_1, \ldots, e_N) = 0] \leq \frac{N}{|S|} \ .$$

Permutation-Matrix Test [TW10]

- Let M be an $N \times N$-matrix over $\mathbb{Z}_q$

$$M = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix}$$

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- Define N-degree polynomial $f_M \in \mathbb{Z}_q[x_1, \ldots, x_N]$ by

$$f_M(x_1, \ldots, x_N) = \prod_i \langle m_i, \overline{x} \rangle = \prod_i \left(\sum_j m_{i,j} x_j \right)$$

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Theorem

M is a permutation matrix iff

$$f_M(\overline{x}) = \prod_j x_j \quad \textit{and} \quad M \cdot \overline{1} = \overline{1}$$

Committed Permutation-Matrix Test

1. P computes "homomorphic" commitment $C(M)$ of permutation matrix M.

2. V chooses $e_1, \ldots, e_N \in S$ randomly and computes

$$C(M) \cdot \overline{e} = C(M \cdot \overline{e}) = \left(C(\langle m_1, \overline{e} \rangle), \ldots, C(\langle m_N, \overline{e} \rangle)\right)$$

3. P proves (knowledge) using standard Schnorr-proofs that

$$f_M(e_1, \ldots, e_N) = \prod_j e_j \quad \text{and} \quad C(M) \cdot \overline{1} = C(\overline{1})$$

Commitment-Consistent Proof of a Shuffle

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3. P proves, using Schnorr-proofs, knowledge of e'_i such that

$$C(M) \cdot \overline{e} = \left(C(e'_1), \ldots, C(e'_N)\right) \quad \text{and} \quad u' = \prod_i (u'_i)^{e'_i}$$

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Submission Schemes

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Must allow extracting the plaintexts of corrupt senders without secret key of cryptosystem. Many ways with pros and cons:

- Interactive zero-knowledge proof of knowledge.
- Non-interactive proof using Fiat-Shamir heuristic.
- Non-interactive zero-knowledge proof (of knowledge).
- Naor-Yung paradigm with "proof" of equality of plaintexts.
- Verifiable secret sharing based "proof of knowledge".
- Based on hashproofs and stripping Cramer-Shoup ciphertexts.
- By construction, e.g., Pret-a-Votér or trusted encryption machine.

Questions?